Exercise 4

Show that Eq. A.2-6 is valid for the particular case $i = 1$, $h = 2$.
Show that Eq. A.2-7 is valid for the particular case $i = j = m = 1$, $n = 2$.

Solution

Eq. A.2-6 says that

$$\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{hjk} = 2\delta_{ih},$$

where δ_{ih} is the Kronecker delta function and ε_{ijk} is the permutation symbol.

$$\delta_{ih} = \begin{cases} 1 & i = h \\ 0 & i \neq h \end{cases}$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, \text{ or } 312 \\ -1 & \text{if } ijk = 321, 132, \text{ or } 213 \\ 0 & \text{if any indices are the same} \end{cases}$$

If $i = 1$ and $h = 2$, then the right-hand side becomes

$$2\delta_{ih} = 2\delta_{12} = 0.$$

The left-hand side becomes

$$\begin{aligned}
\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{hjk} = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{1jk}\varepsilon_{2jk} &= \sum_{j=1}^{3}(\varepsilon_{1j1}\varepsilon_{2j1} + \varepsilon_{1j2}\varepsilon_{2j2} + \varepsilon_{1j3}\varepsilon_{2j3}) \\
&= \sum_{j=1}^{3} \varepsilon_{1j1}\varepsilon_{2j1} + \sum_{j=1}^{3} \varepsilon_{1j2}\varepsilon_{2j2} + \sum_{j=1}^{3} \varepsilon_{1j3}\varepsilon_{2j3} \\
&= \varepsilon_{111}\varepsilon_{211} + \varepsilon_{121}\varepsilon_{221} + \varepsilon_{131}\varepsilon_{231} \\
&\quad + \varepsilon_{112}\varepsilon_{212} + \varepsilon_{122}\varepsilon_{222} + \varepsilon_{132}\varepsilon_{232} \\
&\quad + \varepsilon_{113}\varepsilon_{213} + \varepsilon_{123}\varepsilon_{223} + \varepsilon_{133}\varepsilon_{233} \\
&= 0.
\end{aligned}$$

Both sides are the same. Therefore, Eq. A.2-6 is valid for the particular case $i = 1$, $h = 2$.

Eq. A.2-7 says that

$$\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}.$$

If $i = 1$, $j = 1$, $m = 1$, and $n = 2$, then the right-hand side becomes

$$\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} = \delta_{11}\delta_{12} - \delta_{12}\delta_{11} = 0.$$

The left-hand side becomes

$$\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{mnk} = \sum_{k=1}^{3} \varepsilon_{11k}\varepsilon_{12k} = \varepsilon_{111}\varepsilon_{121} + \varepsilon_{112}\varepsilon_{122} + \varepsilon_{113}\varepsilon_{123} = 0.$$

Both sides are the same. Therefore, Eq. A.2-7 is valid for the particular case $i = j = m = 1$, $n = 2$.